

To the theory of q -ary Steiner and other-type trades*

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Abstract

We introduce the concept of a clique bitrade, which generalizes several known types of bitrades, including latin bitrades, Steiner $T(k-1, k, v)$ bitrades, extended 1-perfect bitrades. For a distance-regular graph, we show a one-to-one correspondence between the clique bitrades that meet the weight-distribution lower bound on the cardinality and the bipartite isometric subgraphs that are distance-regular with certain parameters. As an application of the results, we find the minimum cardinality of q -ary Steiner $T_q(k-1, k, v)$ bitrades and show a connection of minimum such bitrades with dual polar subgraphs of the Grassmann graph $J_q(v, k)$.

Keywords: bitrades, trades, Steiner systems, subspace designs

1. Introduction

In this paper, we prove some results on a rather general class of combinatorial bitrades, mainly concentrating on bitrades of minimum possible cardinality, and obtain a partial result concerning minimum q -ary Steiner bitrades.

As a quick-start introduction to the terminology used in this paper, we consider it by the example of the Steiner triple systems, well-known in combinatorics. Let T be the set of 3-subsets, called triples, of a given finite set V of cardinality $v \geq 6$. We consider a graph on the vertex set T , where two triples are adjacent if and only if they intersect in two elements (this graph is known as the Johnson graph $J(n, 3)$). In this graph, the triples that include a given 2-subset form a maximal clique (a set of mutually adjacent vertices). A subset of T that intersects with every such clique in exactly one vertex is known as a Steiner triple system, or $\text{STS}(v)$, or $\text{S}(2, 3, v)$. A *Steiner bitrade* (of type $T(2, 3, v)$) is defined as a pair (T_0, T_1) of disjoint nonempty subsets (called *trades*) of T such that every 2-subset of V is either included in exactly one triple from T_0 and exactly one triple from T_1 or is not included in any triple of $T_0 \cup T_1$. Equivalently, every maximum clique intersects with each of T_0, T_1 in exactly one element or does not intersect with both of them. Given two different $\text{STS}(v)$ S and S' , the difference pair $(S \setminus S', S' \setminus S)$ is necessarily a Steiner bitrade; however, not every Steiner bitrade can be obtained in such a way. The graph terminology used in this paragraph to treat the Steiner triple systems and the Steiner bitrades allows to define similar structures for other graphs, covering some

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other well-known classes of objects (perfect codes, latin squares and hypercubes, and the corresponding bitrades, see the examples in Section 3), as well as classes that are studied intensively only over the last years (q -ary Steiner systems).

Bitrades (trades) are used in combinatorics, including combinatorial design theory and combinatorial coding theory, to study possible differences between two combinatorial objects from the same class and to obtain new objects from a given one. In particular, small bitrades are used to construct large classes of objects with the same parameters, see e.g. [26], [2], [1], [28], [23], [13]; often minimum trades are utilized to get a lower bound on the number of objects. Trades embedded into “complete” objects are also known under term “switching components” [21]; but in general, bitrades (and trades) are defined independently. Trades can exist even if complete objects with the corresponding parameters do not exist (for example, $v \not\equiv 1, 3 \pmod 6$ for $\text{STS}(v)$). This gives an additional motivation to study trades and, as a result, to develop the theory of “complete” objects of given type.

One interesting and important class of such “complete” objects is the class of subspace designs, which are also known as q -ary generalizations of classic combinatorial designs. In particular, if we replace in the definition of $\text{STS}(v)$ above the set V by a v -dimensional space over $\text{GF}(q)$ and the t -subsets by t -dimensional subspaces, then we get the definition of $\text{STS}_q(v)$, a q -ary Steiner system. It should be mentioned that the increasing interest to subspace codes and subspace designs in the recent years is partially motivated by their importance for network coding applications, see e.g. [16]. Among the series of new results, we mention [4] (nontrivial $\text{STS}_q(v)$ do exist) and [11] (nontrivial q -ary simple t -designs exist for all t). While the trades, in the case of subspace designs, were not considered independently before, an equivalent concept (so-called t -equivalent sets) was used for the construction of subspace designs [5]. The current paper, apart from some general results, which establish common properties of several partial kinds of trades, contains a contribution to the theory of subspace designs. Namely, we find the minimum possible cardinality of a q -ary Steiner trade of type $T_q(d-1, d, n)$.

In Section 2, we define the main notations and concepts (Subsection 2.1), including the concept of a clique bitrade, and prove four general theorems. Theorem 1 (Subsection 2.2) shows that the existence of a clique bitrade in a regular graph is equivalent to the existence of an eigenfunction with certain restrictions and to the existence of a bipartite regular subgraph of certain degree. In Subsection 2.3, we recall the concepts of Delsarte cliques and Delsarte pairs and establish some useful properties of the Delsarte cliques. As a corollary, we prove an intersecting characterization of the eigenfunctions with the minimum eigenvalue, related to these concepts (Theorem 2). In Subsection 2.4, we consider the weight-distribution lower bound on the number of nonzeros of an eigenfunction. Theorem 3 in Subsection 2.5 shows that in the case of a distance-regular graph the existence of a clique bitrade meeting the weight-distribution lower bound is equivalent to the existence of a bipartite regular isometric subgraph of certain degree. Theorem 4 states that the isometric subgraph mentioned above is distance-regular.

In Section 3 we illustrate the theory by examples of bitrades already known in the literature, including ones from design theory (Example 3), coding theory (Example 4), the theory of latin squares and latin hypercubes (Example 2).

In Section 4, based on the results of Section 2 and known facts about the dual polar

graphs, we find q -ary Steiner $T_q(d-1, d, n)$ bitrades of minimum cardinality.

2. General theory

2.1. Basic definitions

Given a connected graph Γ , by the *distance* $d_G(x, y)$ between two vertices x and y , we mean the length of a shortest path from x to y . For a graph $\Gamma = (V, E)$ and a vertex $x \in V$ or a set of vertices $x \subset V$, $\Gamma_i(x)$ denotes the i th neighborhood of x , that is, the set of vertices at distance i from x . The *diameter* $D(\Gamma)$ of Γ is the maximum distance between two vertices of Γ .

An *eigenfunction* of a graph $\Gamma = (V, E)$ is a function $f : V \rightarrow \mathbb{R}$ that is not constantly zero and satisfies

$$\sum_{y \in \Gamma_1(x)} f(y) = \theta f(x) \quad (1)$$

for all x from V and some constant θ , which is called an *eigenvalue* of Γ . The eigenfunctions of a graph can be treated as the eigenvectors of its adjacency matrix.

A set C of vertices of a regular graph Γ of degree k is said to be *completely regular* with *covering radius* ρ if $\Gamma_\rho(C) \neq \emptyset = \Gamma_{\rho+1}(C)$ and there is a sequence $(b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho)$, which is named the *intersection array*, such that $\Gamma_{i+1}(C) \cap \Gamma_1(y) = b_i$ and $\Gamma_{i-1}(C) \cap \Gamma_1(y) = c_i$ hold for every $i \in \{0, \dots, \rho\}$, and every $y \in \Gamma_i(C)$, where $b_\rho = c_0 = 0$. The numbers $b_0, \dots, b_\rho, c_0, \dots, c_\rho$, and a_0, \dots, a_ρ , where $a_i = k - b_i - c_i$, are referred to as the *intersection numbers*, and the tridiagonal matrix $(a_{i,j})_{i,j=0}^\rho$, where $a_{i,i} = a_i$, $a_{i,i+1} = b_i$, $a_{i,i-1} = c_i$, is called the *intersection matrix* of C . By the eigenvalues of a completely regular set, we will mean the eigenvalues of its intersection matrix. Given a completely regular set C of covering radius ρ and one of its eigenvalues θ , by δ_C^θ we denote the function on the vertex set that equals ν_i on $\Gamma_i(C)$, where $(1 = \nu_0, \nu_1, \dots, \nu_\rho)$ is an eigenvector of the intersection matrix corresponding to the eigenvalue θ (it is easy to see for a tridiagonal matrix with nonzero lower- and upper-diagonal elements that any eigenvector is uniquely determined by its first element and the eigenvalue, which means that for each eigenvalue there is a unique eigenvector starting with 1; in particular, there are $\rho + 1$ different eigenvalues). It is straightforward that δ_C^θ is an eigenfunction of the graph with the same eigenvalue θ , which proves the known fact that an eigenvalue of a completely regular set is necessarily an eigenvalue of the graph.

A connected graph Γ is called *distance-regular* if every singleton is completely regular with the same intersection array (independent on the choice of the vertex), which is called the intersection array of Γ .

Let Γ be a connected regular graph of degree k . Assume that S is a set of $(s+1)$ -cliques in Γ such that every edge of Γ is included in exactly m cliques from S ; in this case, we will say that the pair (Γ, S) is a (k, s, m) *pair*. A couple (T_0, T_1) of mutually disjoint nonempty sets of vertices is called an S -*bitrade*, or a *clique bitrade*, if every clique from S either intersects with each of T_0 and T_1 in exactly one vertex or does not intersect with both of them (in particular, this means that each of T_0, T_1 is an independent set in Γ).

A set of vertices T_0 is called an S -trade if there is another set T_1 (known as a *mate* of T_0) such that the pair (T_0, T_1) is an S -bitrade.

2.2. A bitrade criterion

We start with a criterion, which can be used as alternative definition of a clique bitrade.

Theorem 1. *Let Γ be a regular graph of degree k . Let (Γ, S) , where S is a set of cliques in Γ , be a (k, s, m) pair. Let $T = (T_0, T_1)$ be a pair of disjoint nonempty independent sets of vertices of Γ . The following assertions are equivalent.*

- (a) T is an S -bitrade.
- (b) The function

$$f^T(x) = \begin{cases} (-1)^i & \text{if } \bar{x} \in T_i, i \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

is an eigenfunction of Γ with eigenvalue $\theta = -k/s$.

- (c) The subgraph Γ^T of Γ generated by the vertex set $T_0 \cup T_1$ is regular with degree $-\theta = k/s$ (as T_0 and T_1 are independent sets, this subgraph is bipartite).

PROOF. All proofs are based on counting arguments, mainly utilizing the definition of a (k, s, m) pair.

(a) \Rightarrow (b): Assume (T_0, T_1) is an S -bitrade.

At first, we show that (1) holds for every vertex $x \notin T_0 \cup T_1$. Indeed, double-counting the number of triples (u, t_0, t_1) such that $t_0, t_1, x \in u \in S$ and $t_l \in \Gamma_1(x) \cap T_l$ gives

$$|\Gamma_1(x) \cap T_0| \cdot m = |\Gamma_1(x) \cap T_1| \cdot m.$$

Thus, (1) holds with both sides being equal to zero.

It remains to prove (1) for $x \in T_0$ (the case $x \in T_1$ is similar). Double-counting the number of triples (u, t, t_1) such that $t, t_1, x \in u \in S$, $t \in \Gamma_1(x) \setminus T_1$ and $t_1 \in \Gamma_1(x) \cap T_1$ gives

$$|\Gamma_1(x) \cap T_1| \cdot m \cdot (s - 1) = |\Gamma_1(x) \setminus T_1| \cdot m \cdot 1$$

(for the left side, we choose t_1 first then u containing x and t_1 , and finally t from $u \setminus \{x, t_1\}$; the right side corresponds to the order t, u, t_1). This implies $|\Gamma_1(x) \cap T_1| = |\Gamma_1(x)|/s = k/s$. Since T_0 is independent and therefore we have $|\Gamma_1(x) \cap T_0| = 0$, we find that (1) turns to $\theta = \theta$.

(b) \Rightarrow (c) is trivial.

(c) \Rightarrow (a): Let Γ^T be regular of degree k/s . Let us consider some x from T_0 . There are km/s cliques from S containing x (as well as any other fixed vertex). On the other hand, every $y \in \Gamma_1(x) \cap T_1$ is in m of them. Since, by the hypothesis, there are k/s such y , every clique from S containing x contains some $y \in \Gamma_1(x) \cap T_1$. By the definition of an S -bitrade, the claim follows. \blacktriangle

2.3. Delsarte cliques, Delsarte pairs, and eigenfunctions

We can say more if Γ is distance-regular. It is known [14] (see also [6, Proposition 4.4.6]) that a clique in a distance-regular graph cannot have more than $1 - k/\theta_{\min}$ elements, where θ_{\min} is the minimum eigenvalue of the graph; the cliques of cardinality $1 - k/\theta_{\min}$ are called *Delsarte cliques*. A (k, s, m) pair (Γ, S) is known as a *Delsarte pair* if the graph Γ is distance-regular and S consists of Delsarte cliques [3].

Proposition 1. *If, under notation and hypothesis of Theorem 1, (a)–(c) hold and, additionally, the graph Γ is distance-regular, then θ is the minimum eigenvalue of Γ , $s + 1$ is the maximum order of a clique in Γ , and (Γ, S) is a Delsarte pair.*

PROOF. As proved in [14], any clique of order M satisfies $M \leq 1 - k/\theta_{\min}$, where θ_{\min} is the minimum eigenvalue of Γ . From (b) we have $s + 1 = 1 - k/\theta$. Thus, $\theta = \theta_{\min}$ and $s + 1$ is the maximum order of a clique in Γ . The pair (Γ, S) is a Delsarte pair by the definition. \blacktriangle

We will use the following properties of the Delsarte cliques:

Proposition 2. *Let Γ be a distance-regular graph of degree k and diameter D , and let θ be the minimum eigenvalue of Γ . Then*

- (i) *every Delsarte clique is a completely-regular set of covering radius $D - 1$, and θ is not one of its eigenvalues;*
- (ii) *the sum of any eigenfunction with eigenvalue θ over a Delsarte clique is zero;*
- (iii) *there are positive numbers $s_0^+, \dots, s_{D-1}^+, s_1^-, \dots, s_D^-$ such that for every vertex x and every Delsarte clique C at distance i from x , there hold $|\Gamma_i(x) \cap C| = s_i^+$ and $|\Gamma_{i+1}(x) \cap C| = s_{i+1}^-$.*

PROOF. It is shown in [6, Proposition 4.4.6, Remark] that a Delsarte clique is a completely-regular set and its covering radius ρ is less than D ; hence, $\rho = D - 1$. The second claim of (i) can also be retrieved from the proof of [6, Proposition 4.4.6] and some algebraic background, but we will give another proof. Let $\theta_0, \dots, \theta_{D-1}, \theta'$ be the eigenvalues of Γ , where θ' is not an eigenvalue of the given Delsarte clique C . Then the characteristic function χ_C of C is a linear combination of $\delta_C^{\theta_0}, \dots, \delta_C^{\theta_{D-1}}$ (as well as the vector $(1, 0, \dots, 0)$ is a linear combination of the eigenfunctions of the intersection matrix of C). Thus, χ_C is orthogonal to any eigenfunction with eigenvalue θ' ; in other words, the sum of the values of any eigenfunction with eigenvalue θ' over C is zero. Let us consider the eigenfunction $\delta_{\{x\}}^{\theta'}$ for some x from C . We have $\delta_{\{x\}}^{\theta'}(x) = 1$ and $\delta_{\{x\}}^{\theta'}(y) = \theta'/k$ for all y from $C \setminus \{x\}$. Since, by the definition of a Delsarte clique, $|C \setminus \{x\}| = -k/\theta$, we have $1 + (k/\theta')(-k/\theta) = 0$. So, $\theta' = \theta$, and (i) and (ii) hold.

(iii) Let $\nu^\theta = (\nu_0, \nu_1, \dots, \nu_D)$ be the eigenvector of the intersection matrix of Γ corresponding to θ and starting with 1. Let us consider a vertex x at distance i from the given Delsarte clique C . Since the sum of the values of $\delta_{\{x\}}^\theta$ over C is zero, we have

$$|\Gamma_i(x) \cap C| \nu_i + |\Gamma_{i+1}(x) \cap C| \nu_{i+1} = 0. \quad (3)$$

If the first summand of the left part is nonzero, then the second one is nonzero too; it follows by induction on i that all ν_i , $i = 0, 1, \dots, D$, are nonzero. Then, (3) implies that $|\Gamma_i(x) \cap C|/|\Gamma_{i+1}(x) \cap C| = -\nu_{i+1}/\nu_i$. Claim (iii) follows. \blacktriangle

As a corollary, for the distance-regular graphs, we can formulate a stronger analog of the equivalence of (a) and (b) in Theorem 1:

Theorem 2. *Let (Γ, S) be a Delsarte pair and θ be the minimum eigenvalue of Γ . Then*

- (i) *A function f over the vertex set of Γ is an eigenfunction with the eigenvalue θ if and only if for every clique C from S it holds $\sum_{x \in C} f(x) = 0$.*
- (ii) *A proper subset B of the vertex set of Γ is a completely regular set of radius 1 with eigenvalue θ if and only if it has a constant number of elements in any clique from B .*

PROOF. In (i), the “if” statement follows from direct checking of (1), while “only if” comes from Proposition 2(ii). In (ii), again, “if” is straightforward, while “only if” follows from claim (i) if we consider the eigenfunction δ_B^θ . \blacktriangle

2.4. The weight-distribution bound

The rest of Section 2 is devoted to the bitrades that are minimum in the sense that their cardinality meets a special lower bound. In this subsection, we define the weight distribution and the weight-distribution bound.

By the *weight distribution* of a function $f : V \rightarrow \mathbb{R}$ with respect to a vertex x of a graph $\Gamma = (V, E)$ we will mean the sequence $W(x) = (W^i(f))_{i=0}^{D(\Gamma)}$, where $W^i(f) = \sum_{y \in \Gamma_i(x)} f(y)$. The following fact is well known and easy to derive from definitions, by induction on i .

Lemma 1. *The weight distribution $W(x)$ of an eigenfunction f of a distance-regular graph Γ is calculated as $(f(x)W_{A,\theta}^i)_{i=0}^{D(\Gamma)}$ where the coefficients $W_{A,\theta}^i$ are derived from the intersection array $A = (b_0, \dots, c_{D(\Gamma)})$ of Γ and the eigenvalue θ that corresponds to f :*

$$W_{A,\theta}^0 = 1, \quad W_{A,\theta}^1 = \theta, \quad W_{A,\theta}^i = ((\theta - a_{i-1})W_{A,\theta}^{i-1} - b_{i-2}W_{A,\theta}^{i-2})/c_i, \quad i \geq 2. \quad (4)$$

To read more about how to calculate the weight distribution of eigenfunctions and generalizations of eigenfunctions, see [17]. Using Lemma 1, it is easy to derive the following lower bound on the support of an eigenfunction. The bound is also known; a partial case of this argument was used in [10] to find the minimum cardinality of a switching component of binary 1-perfect codes (which can also be treated in terms of eigenfunctions).

Corollary 1 (the weight-distribution (w.d.) bound). *An eigenfunction f of a distance-regular graph has at least $\sum_{i=0}^{D(\Gamma)} |W_{A,\theta}^i|$ nonzeros, in notation of Lemma 1.*

PROOF. Considering the weight distribution with respect to a vertex x with the maximum value of $|f(x)|$, we see that the number of nonzeros in $\Gamma_i(x)$ is at least $|W_{A,\theta}^i|$. \blacktriangle

We will say that an eigenfunction of a distance-regular graph (and the corresponding clique bitrade, if any) *meets the w.d. bound* if it has exactly $\sum_{i=0}^{D(\Gamma)} |W_{A,\theta}^i|$ nonzeros.

2.5. Bitrades that meet the w.d. bound

In this section, we are focused on minimum clique bitrades in distance-regular graphs.

Theorem 3. *Let Γ be a distance-regular graph of degree k . Under notation and hypothesis of Theorem 1, the following assertions are equivalent.*

- (a') T is an S -bitrade meeting the w.d. bound.
- (b') The function f^T is an eigenfunction of Γ meeting the w.d. bound with eigenvalue $-k/s$.
- (c') The subgraph Γ^T is a regular isometric subgraph with degree k/s .

PROOF. (a') \Leftrightarrow (b') is straightforward from (a) \Leftrightarrow (b) of Theorem 1 and the definition of the concept "to meet the w.d. bound."

(c') \Rightarrow (b'). Assume (c') holds. Consider some x from T_0 . By the isometry property,

$$\Gamma_i(x) \cap (T_0 \cup T_1) = \Gamma_i^T(x) \cap (T_0 \cup T_1) = \Gamma_i^T(x) \cap T_{i \bmod 2}$$

(the last equality holds because Γ^T is bipartite with parts T_0 and T_1). It follows that f^T is either non-negative or non-positive on $\Gamma_i(x)$ and $|W^i(f^T)| = |\Gamma_i(x) \cap (T_0 \cup T_1)|$. Thus, $|T_0 \cup T_1| = \sum_{i=0}^{D(\Gamma)} |W^i(f^T)|$, and f^T meets the w.d. bound.

(a', b') \Rightarrow (c'). Assume (b') holds. Then for every $x \in T_0 \cup T_1$ and for every i , the function f^T is either non-negative, or non-positive on $\Gamma_i(x)$. That is, $\Gamma_i(x)$ does not intersect with either T_0 or T_1 . Let us prove by induction on $d_\Gamma(x, y)$ that

$$d_{\Gamma^T}(x, y) = d_\Gamma(x, y)$$

for every $x, y \in T_0 \cup T_1$. For $d_\Gamma(x, y) = 0$, this is trivial. Let $x, y \in T_0$ and $d_\Gamma(x, y) = i$ (the case when x , or y , or both belong to T_1 is similar). There is a vertex v in $\Gamma_{i-1}(x) \cap \Gamma_1(y)$. A clique from S that contains both v and x has a vertex z from T_1 . All the vertices of this clique lie in $\Gamma_{i-1}(x) \cup \Gamma_i(x)$. But z cannot belong to $\Gamma_i(x)$ as $\Gamma_i(x)$ already contains a vertex from T_0 . Hence, $z \in \Gamma_{i-1}(x)$. By the induction hypothesis, $d_{\Gamma^T}(x, z) = i - 1$. Therefore, $d_{\Gamma^T}(x, y) = (i - 1) + 1 = d_\Gamma(x, y)$, which proves the statement. \blacktriangle

Theorem 4. Assume that, under the notation and the hypothesis of Theorem 3, (a')–(c') hold. Then the graph Γ^T is distance-regular. Moreover, for every vertex x of Γ^T and every i it holds $|\Gamma_i^T(x)| = |W_{A,\theta}^i|$ where $W_{A,\theta}^i$ satisfies (4) with the intersection numbers of Γ .

PROOF. Consider vertices x and y of Γ^T at distance i from each other. Without loss of generality we assume $y \in T_0$. We know that the cardinality of $\Gamma_{i+1}(x) \cap \Gamma_1(y)$ is b_i , the corresponding intersection number of Γ . Every vertex from this set is in m cliques of S containing y . On the other hand, every such clique contains exactly s_{i+1}^- vertices of $\Gamma_{i+1}(x) \cap \Gamma_1(y)$. So, the number of cliques containing y and intersecting with $\Gamma_{i+1}(x)$ is mb_i/s_{i+1}^- . Each such clique contains one element from T_1 , and this element lies in $\Gamma_{i+1}(x)$, because the considered bitrade meets the w.d. bound. On the other hand, every such element belongs to m cliques containing y . So, $|\Gamma_{i+1}(x) \cap \Gamma_1(y) \cap T_1| = (mb_i/s_{i+1}^-)/m = b_i/s_{i+1}^-$. By the isometry property, we have $|\Gamma_{i+1}^T(x) \cap \Gamma_1^T(y)| = b_i/s_{i+1}^-$. Similarly, $|\Gamma_{i-1}^T(x) \cap \Gamma_1^T(y)| = c_i/s_{i-1}^+$, and the graph Γ^T is distance-regular by the definition.

The last statement of the theorem follows from Theorem 3(b') and the proof of Corollary 1. \blacktriangle

Corollary 2. For every distance-regular graph Γ admitting a Delsarte pair, there is a sequence $A' = (b'_0, \dots, b'_{D(\Gamma)-1}; c'_1, \dots, c'_{D(\Gamma)})$ such that the existence of a clique bitrade meeting the w.d. bound in Γ is equivalent to the existence of an isometric distance-regular subgraph with intersection array A' .

In the following sections, we will consider examples of such subgraph.

3. Known examples

By a *clique design*, we mean a set of vertices that has exactly one vertex in common with each clique of S , given a Delsarte pair (Γ, S) . The difference couple $(D_1 \setminus D_2, D_2 \setminus D_1)$ of two different clique designs is always a clique bitrade, while the existence of a clique trade does not imply the existence of a clique design in the same graph. In this section, we will consider classes of distance-regular graphs for which the theory of clique designs and clique trades, in different notations, is more-or-less developed in the corresponding areas of mathematics.

Example 1. We start with a very simple example, when the graph is an n -dimensional octahedron, a regular graph with $2n$ vertices of degree $2n - 2$. There are 2^n maximum cliques of cardinality n ; a clique design consists of two non-adjacent vertices; a minimum bitrade corresponds to a square subgraph. A less trivial problem is to characterize all (n, m) systems of cliques (for different m). One can find that such systems are in one-to-one correspondence with the Boolean functions with $4m$ ones whose correlation immunity [24] is at least 2.

Example 2. The vertex set of the *Hamming graph* $H(n, q)$ is the set $\{0, \dots, q - 1\}^n$ of words of length n over the alphabet $\{0, \dots, q - 1\}$. The graph $H(n, 2)$ is also known as the n -cube, or the *hypercube* of dimension n . Two words are adjacent whenever they differ in exactly one position. The clique designs in Hamming graphs are known as the *latin hypercubes* (in coding theory, these objects are known as the *distance-2 MDS codes*), and the clique bitrades, as the *latin bitrades* [22]. The most studied case, which corresponds to the latin squares, is $n = 3$, see e.g. [7]. The graph corresponding to a minimum bitrade is $H(n, 2)$ [22].

Example 3. The vertices of the *Johnson graph* $J(n, w)$ are the w -subsets of a given set N of cardinality n . Two different vertices are adjacent whenever they intersect in $w - 1$ elements. The graphs $J(n, w)$ and $J(n, n - w)$ are isomorphic, and below we assume $2w \leq n$. A *Steiner* $S(w - 1, w, n)$ *system* S is defined as a set of vertices of $J(n, w)$, usually called *blocks*, such that every $(w - 1)$ -subset of N is included in exactly one block from S (see e.g. [9]). It is easy to see that the set of w -subsets of N that include a given $(w - 1)$ -subset is a maximum clique in $J(n, w)$. So, the Steiner $S(w - 1, w, n)$ systems are the clique designs in $J(n, w)$. The clique bitrades in $J(n, w)$ are known as the *Steiner* $T(w - 1, w, n)$ *bitrades*, (in an alternative terminology, Steiner $T(w - 1, w, n)$ *trades*) see e.g. [12]. Any minimum bitrade has the form

$$\left(\left\{ \{a_1^{b_1}, \dots, a_w^{b_w}\} \mid b_1, \dots, b_w \in \{0, 1\}, b_1 + \dots + b_w \equiv 0 \pmod{2} \right\}, \right. \\ \left. \left\{ \{a_1^{b_1}, \dots, a_w^{b_w}\} \mid b_1, \dots, b_w \in \{0, 1\}, b_1 + \dots + b_w \equiv 1 \pmod{2} \right\} \right),$$

where $a_1^0, \dots, a_w^0, a_1^1, \dots, a_w^1$ are distinct elements of N . The corresponding subgraph is $H(w, 2)$. The minimum bitrade cardinality was found in [15]. In the case $w = 3$, the minimum trade is known as the Pasch configuration, or the quadrilateral.

Example 4. The vertices of the *halved n -cube* are the even-weight binary words of length n (i.e., a part of the bipartite n -cube). Two words are adjacent whenever they differ in exactly two positions. A maximum clique is the set of binary n -words adjacent in $H(n, 2)$

to a fixed odd-weight word; such clique is Delsarte if and only if n is even. The clique designs in halved n -cubes are the *extended 1-perfect codes*. Such codes exist if and only if n is a power of two, see e.g. [19]. The minimum cardinality $2^{n/2}$ of a bitrade was found in [10] (the authors considered a special type of 1-perfect trades, but the argument works for the general case; the 1-perfect trades in $H(n-1, 2)$ are in one-to-one correspondence with the extended 1-perfect trades in the halved n -cube). An example of a minimum clique bitrade is $\{(x, x) \mid x \in \{0, 1\}^{n/2}\}$. The graph corresponding to a minimum bitrade is $H(n/2, 2)$.

Example 5. If for every vertex x of a distance-regular graph Γ , there is exactly one vertex y at distance $d = D(\Gamma)$ from x , then identifying all such pairs x, y results in a distance-regular graph of diameter $\lfloor d/2 \rfloor$, known as the *folded* Γ . It is not difficult to see that the bipartite isometric subgraph Γ^T corresponding to a minimum clique bitrade will be also folded under this operation. However, the folded Γ^T is bipartite if and only if d is odd; this reflects the fact that the minimum eigenvalue of Γ is an eigenvalue of the folded Γ if and only if d is even. Examples are the folded $J(2d, d)$ and the folded halved $H(2d, 2)$, where the corresponding subgraph is the folded $H(d, 2)$, d is even.

The next example shows that analogs of the clique bitrades can be considered even if the graph has no cliques of required cardinality.

Example 6. The Shrikhande graph can be defined on the 16 quaternary pairs from \mathbb{Z}_4^2 , where two pairs are adjacent if and only if their element-wise difference is one of $(0, 1)$, $(0, 3)$, $(1, 0)$, $(3, 0)$, $(1, 1)$, $(3, 3)$. The Doob graph $D(m, n)$ is the Cartesian product of $m > 0$ copies of the Shrikhande graph and n copies of the complete graph on 4 vertices. This graph is distance regular with the same intersection array as the Hamming graph $H(2m + n, 4)$. It follows that it has the same minimum eigenvalue $\theta = -2m - n$, and the w.d. bound on the number of nonzeros of an eigenfunction is the same too, i.e., 2^{2m+n} , for θ . However, the Doob graph does not admit a Delsarte pair; moreover, Delsarte cliques, which have cardinality 4, does not occur in $D(m, 0)$. So, we cannot apply the definition of a clique design. Nevertheless, we can apply an alternative definition using Theorem 1(b): let us say that a pair of two disjoint independent vertex sets is a pseudo-clique bitrade if the difference of their characteristic functions is an eigenfunction with minimum eigenvalue. An example of a minimum bitrade is $\{(0, 0), (0, 1), (0, 2), (0, 3)\}^m \{0, 1\}^n$; it is not difficult to find that the subgraph generated by any minimum bitrade is $H(2n + m, 2)$. In a same manner, a pseudo-clique design can be defined as an independent completely regular set with minimum eigenvalue and covering radius 1. Such sets are the maximum independent sets in the Doob graph [18]; we leave constructing an example as an exercise.

From the last example, we see that defining bitrades in terms of eigenfunctions is a more general approach than in terms of Delsarte cliques. In a similar manner, bitrades with other eigenvalues can be defined. For example, the bitrades with eigenvalue -1 (1-perfect bitrades) are studied in the theory of 1-perfect codes, see e.g. [27].

Remark 1. One can weaken the notion of a *clique design* by defining it as a set that intersects with every clique from S in a constant λ number of elements, not necessarily $\lambda = 1$. In spite of weakening the definition, a clique design is still a completely regular set, but it is not an independent set if $\lambda > 1$. In the partial cases corresponding to the

considered examples (as well as to the case considered in the next section), such designs are also studied in the literature.

4. Minimum q -ary Steiner bitrades

Let F_q^n be an n -dimensional vector space over the Galois field F_q of prime-power order q . The *Grassmann graph* $J_q(n, d)$ is defined as follows. The vertices are the d -dimensional subspaces of F_q^n . Two vertices are adjacent whenever they intersect in a $(d - 1)$ -dimensional subspace. The Grassmann graph is a distance-transitive graph of degree $q \begin{bmatrix} d \\ 1 \end{bmatrix}_q \begin{bmatrix} n-d \\ 1 \end{bmatrix}_q$, where $\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=0}^{b-1} \frac{q^{a-i}-1}{q^{i+1}-1}$ see e.g. [6, Theorem 9.3.3].

All vertices that include a fixed $(d - 1)$ -dimensional subspace form a clique of order $M = \begin{bmatrix} n-d+1 \\ 1 \end{bmatrix}_q$ in $J_q(n, d)$; if $n \geq 2d$ then this clique is maximum. We form an $(M, 1)$ system S from all cliques that correspond to a $(d - 1)$ -dimensional subspace. A set of vertices that intersects with every clique from S in exactly one vertex is known as a q -ary Steiner $S_q(d - 1, d, n)$ system. Constructing q -ary Steiner $S_q(d - 1, d, n)$ systems with $d \geq 3$ is not easy; at the moment, only the existence of $S_2(2, 3, 13)$ is known in this field [4]. An S -bitrade in $J_q(n, d)$ is called a *Steiner $T_q(d - 1, d, n)$ bitrade*.

Before formulating the main theorem of this section, we briefly introduce the dual polar graph $D_d(q)$ (see, e.g., [6]), which plays the role of the bipartite subgraph Γ^T for $\Gamma = J_q(n, d)$. Note that the class of dual polar graphs contains several other subclasses [6, §9.4], which are not considered here, but the graphs of type $D_d(q)$ are the only dual polar graphs that are bipartite.

A quadratic form $Q : F_q^n \rightarrow F_q$ is said to be *nondegenerate* if its kernel $\{x \mid Q(y + x) = Q(y) \forall y \in F_q^n\}$ is zero. A subspace V of F_q^n is called *totally isotropic* whenever the form vanishes completely on V , i.e., $Q(V) = \{0\}$. The maximum dimension of a totally isotropic subspace is known as the *Witt index* of Q . If $n = 2d$, then the maximum Witt index of a nondegenerate quadratic form is equal to d . There exists a unique (up to isomorphism) nondegenerate quadratic form with the Witt index d . One of its representations is $Q_0(v_1, \dots, v_d, u_1, \dots, u_d) = v_1 u_1 + \dots + v_d u_d$. The *dual polar graph* $D_d(q)$ has as vertices the d -dimensional totally isotropic subspaces, with respect to Q_0 ; two vertices α and β are adjacent whenever $\dim(\alpha \cap \beta) = d - 1$.

Theorem 5. *The minimum cardinality of a Steiner $T_q(d - 1, d, n \geq 2d)$ bitrade is*

$$\prod_{i=1}^d (q^{d-i} + 1) = \sum_{i=0}^d q^{\binom{i}{2}} \begin{bmatrix} d \\ i \end{bmatrix}_q, \quad (5)$$

which is also equal to the value of the w.d. bound.

The bipartite distance-regular subgraph of $J_q(n, d)$ generated by a Steiner $T_q(d - 1, d, n)$ bitrade has the parameters of the dual polar graph $D_d(q)$.

PROOF. $J_q(2d, d)$ is an isometric subgraph of $J_q(n, d)$; $D_d(q)$ is an isometric subgraph of $J_q(2d, d)$ [6, p.276]. $D_d(q)$ is a bipartite distance-regular graph of degree $(q^d - 1)/(q - 1)$ (the biparticity and the degree are easily retrieved from the intersection array [6, Theorem 9.4.3]) and order $\prod_{i=1}^d (q^{d-i} + 1)$ [6, p.274, Lemma 9.4.1]. A proof of the identity (5) can be

found in [25, Equation (1.87)]. Since $(q^d - 1)/(q - 1) = k(M - 1)$ with $k = q \begin{bmatrix} d \\ 1 \end{bmatrix}_q \begin{bmatrix} n-d \\ 1 \end{bmatrix}_q$ and $M = \begin{bmatrix} n-d+1 \\ 1 \end{bmatrix}_q$, the result follows from Theorem 3. \blacktriangle

Remark 2. It can be found that the i th summand $S_i = q^{\binom{i}{2}} \begin{bmatrix} d \\ i \end{bmatrix}_q$ of the right part of (5) coincides with the number $|D_d(q)_i(x)|$ of vertices at distance i from a fixed vertex x in $D_d(q)$. A straightforward way to prove this is checking the relation $b'_{i-1}S_{i-1} = c'_iS_i$ where $b'_i = q^i \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q$ and $c'_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$ are coefficients from the intersection array of $D_d(q)$, which can be found in [6, Theorem 9.4.3] (this relation correspond to double-counting the edges between $D_d(q)_{i-1}(x)$ and $D_d(q)_i(x)$).

Remark 3. The minimum $T_q(2, 3, n)$ trades can be considered as q -ary analogs of the Pasch configuration (quadrilateral). In particular, the formula $(q + 1)(q^2 + 1) = 15, 40, 85, 156, \dots$ ($q = 2, 3, 4, 5, \dots$) for the size of a minimum trade (which is the half of the size of a minimum bitrade (5)) is satisfied by the Pasch configuration with $q = 1$. As in the case of the Pasch configuration, the graph $J_q(n, 3)$, if n is large enough, contains many isomorphic copies of the minimum $T_q(2, 3, n)$ trade that lie at distance more than 1 from each other (so, simple metrical arguments do not forbid them to belong the same q -ary Steiner system). Indeed, each 6-dimensional subspace of F_q^n corresponds to a subgraph isomorphic to $J_q(6, 3)$, which has a subgraph isomorphic to $D_3(q)$. If two such subspaces have no common 2-dimensional subspace, then the corresponding subgraphs are mutually independent. However, the question how many (0, 1, very few, or good many) minimum trades a real q -ary Steiner system can include remains untouched.

Another representation of the $T_q(2, 3, n)$ trades constructed in the current paper was announced in [20].

Problem 1. *The following question is natural: is a minimum Steiner $T_q(d-1, d, n)$ bitrade unique, up to isomorphism of the Grassmann graph? As noted in [6, Remark 9.4.6], in general, the dual polar graph $D_d(q)$ is not unique as a distance regular graph with given intersection array. The question is if there are nonisomorphic isometric embeddings of such graphs into the Grassmann graph. Note that the minimum trades from the examples of Section 3 are known to be unique.*

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